

ON SOME APPLICATIONS AND SIMULATIONS OF COUNTING PROCESSES

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Abstract: *In this paper we discuss a special case of stochastic processes, namely counting processes. In particular, we describe a discrete-time counting process (Binomial) and a continuous-time one (Poisson). We present the main theoretical characteristics and properties, algorithms for computer simulations of such processes and some interesting applications.*

Keywords: *Binomial counting processes, Poisson counting processes, stochastic processes, computer simulations, MATLAB.*

1. Introduction

Stochastic processes are random variables that change and evolve in time. As such, they play an important role in probability theory and related fields, providing good models for many real-life phenomena. Among them, *counting processes* deal with the number of occurrences of some events in time. As the name says, they count events, such as job arrivals (in a queuing system), message transmissions, customers served, completed tasks, holding times, times of various events occurrence, etc. They have many applications in renewal processes, survival analysis, seismology, software reliability and many other fields (see [1, 3, 4]). In this paper, we discuss Binomial and Poisson counting processes, presenting their main characteristics, some applications and simulations.

1.1 Random Variables

We start with a brief review of basic notions from Probability Theory. Let \mathcal{S} denote the sample space (the set of all possible outcomes) of some experiment and P a probability function (see e.g. [2, 8]).

Definition 1.1. *A random variable is a function $X : \mathcal{S} \rightarrow \mathbb{R}$ for which $P(X \leq x)$ exists, for all $x \in \mathbb{R}$. If $X(\mathcal{S}) \subseteq \mathbb{R}$ is a countable subset of \mathbb{R} , then X is called a discrete random variable, otherwise, it is a continuous random variable.*

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A discrete random variable X is described by its **probability distribution function (pdf)** or **probability mass function (pmf)**, an array that contains all the values taken by it, x_i , and the corresponding probabilities $p_i = P(X = x_i)$,

$$X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I} \quad (1.1)$$

If X is a continuous random variable, we give its **probability density function (pdf)**, i.e. the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = F'(x)$.

We recall some discrete probability laws, which will be used later on.

Bernoulli distribution $Bern(p)$, with parameter $p \in (0,1)$. This is one of the simplest discrete distributions, with pdf

$$X \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}. \quad (1.2)$$

This distribution best models a Bernoulli trial, i.e. the occurrence of “success/failure” in a trial.

Binomial distribution $B(n,p)$, with parameters $n \in \mathbb{N}, p \in (0,1)$. In a series of n Bernoulli trials with success probability p in each trial ($q = 1 - p$), we let X denote the number of successes that occur. Then X has a Binomial distribution, with pdf

$$X \left(C_n^k p^k q^{n-k} \right)_{k=0, \dots, n}. \quad (1.3)$$

It is easy to notice that a Binomial $B(n,p)$ variable is the sum of n independent $Bern(p)$ variables and that $Bern(p) = B(1,p)$.

Geometric distribution $Geo(p)$, with parameter $p \in (0,1)$. In an infinite sequence of Bernoulli trials with success probability p in each trial ($q = 1 - p$), let X denote the number of failures that occur before the first success. Then X has a Geometric distribution, with pdf

$$X \begin{pmatrix} k \\ pq^k \end{pmatrix}_{k=0,1, \dots}. \quad (1.4)$$

Shifted Geometric distribution $SGeo(p)$, with parameter $p \in (0,1)$. In the context described above, this is the number of trials needed to get the first success (as opposed to failures in the previous pdf). If X has a $Geo(p)$ distribution, then $X + 1$ has a Shifted Geometric distribution, hence, the name.

Poisson distribution $Poiss(\lambda)$, with parameter $\lambda > 0$, with pdf

$$X \begin{pmatrix} k \\ \frac{\lambda^k}{k!} e^{-\lambda} \end{pmatrix}_{k=0,1, \dots}. \quad (1.5)$$

A Poisson variable X denotes the number of “rare” events (discrete occurrences of infrequently observed events) that occur in a given interval of time. The parameter

λ represents the average number of such rare events per time unit. The Poisson distribution is used to model number of occurrences of discrete events in an interval of time, such as arrival of jobs (tasks, messages, signals, phone calls, etc.), accidents, earthquakes, that happen in given area, errors found in software, etc.

In what follows, we recall some important continuous distributions. Note that we only mention the expression of the pdf on the region where it is non-zero (so it is understood that it is equal to 0, elsewhere).

Uniform distribution $Unif(a, b)$, with parameters $a < b \in \mathbb{R}$, has pdf

$$f(x) = \frac{1}{b-a}, \quad x \in (a, b). \quad (1.6)$$

It is used when a variable can take any value randomly from an interval, so all values from an interval are equally probable to be taken by that random variable.

Exponential distribution $Exp(\lambda)$ with parameter $\lambda > 0$, has pdf

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0. \quad (1.7)$$

An Exponential variable models time: interarrival time (time between arrival of jobs), halftime, failure time, time between rare events, etc. The parameter λ represents the frequency of rare events, measured in time⁻¹.

It is worth mentioning that in a sequence of rare events, where the number of rare events occurring in an interval of time of length t has a $Poiss(\lambda t)$ distribution, the time between rare events is modeled by an $Exp(\lambda)$ distribution.

1.2 Stochastic processes and Markov chains

Many random variables are not static, they change and develop in time.

Definition 1.3. A *stochastic process* is a random variable that depends on time. It is denoted by $X(t, e)$ or $X_t(e)$, where $t \in \mathcal{T}$ is time and $e \in \mathcal{S}$ is an outcome. The values of $X(t, e)$ are called *states*.

If $t \in \mathcal{T}$ is fixed, then X_t is a random variable, while if we fix $e \in \mathcal{S}$, X_e is a function of time, called a *realization* or *sample path* of the process $X(t, e)$.

Definition 1.4. A *stochastic process* is called *discrete-state* if $X_t(e)$ is a discrete random variable for all $t \in \mathcal{T}$ and *continuous-state* if $X_t(e)$ is a continuous random variable, for all $t \in \mathcal{T}$.

Similarly, a stochastic process is *discrete-time* if the set \mathcal{T} is discrete and *continuous-time* if the set of times \mathcal{T} is an interval (bounded or unbounded) in \mathbb{R} .

In what follows, we omit writing e as an argument of a stochastic process (as it is usually done when writing random variables).

Definition 1.5. A stochastic process X_t is **Markov** if for any times $t_1 < t_2 < \dots < t_n < t$ and any sets A_1, A_2, \dots, A_n, A ,

$$P(X_t \in A \mid X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) = P(X_t \in A \mid X_{t_n} \in A_n). \quad (1.8)$$

Definition 1.6. A **Markov chain** is a discrete-state, discrete-time, Markov stochastic process.

To simplify the notations, we do the following: A Markov chain is a discrete-time process, so we can see it as a sequence of random variable $\{X_0, X_1, \dots\}$, where X_k describes the situation at time $t = k$. Since it is also a discrete-state process, we can denote the states by $1, 2, \dots, n$ (they may start at other value and n may not be finite). Then the random variable X_k has the pdf

$$X_k \begin{pmatrix} 1 & 2 & \dots & n \\ P_k(1) & P_k(2) & \dots & P_k(n) \end{pmatrix}, \quad (1.9)$$

where $P_k(1) = P(X_k = 1), \dots, P_k(n) = P(X_k = n)$. Then the Markov property (1.8) can be written as follows:

$$P(X_{t+1} = j | X_t = i, X_{t-1} = l, \dots) = P(X_{t+1} = j | X_t = i), \text{ for all } t \in \mathcal{T}. \quad (1.10)$$

All this information is summarized in a matrix.

Definition 1.7.

– The probabilities

$$p_{ij}(t) = P(X_{t+1} = j | X_t = i), p_{ij}^{(h)}(t) = P(X_{t+h} = j | X_t = i) \quad (1.11)$$

are called the **transition and h-step transition probability**, respectively.

– The matrices

$$P(t) = [p_{ij}(t)]_{i,j=\overline{1,n}}, P^{(h)}(t) = [p_{ij}^{(h)}(t)]_{i,j=\overline{1,n}} \quad (1.12)$$

are called the **transition and h-step transition probability matrix**, respectively, at time t .

Definition 1.8. A Markov chain is said to be **homogeneous (stationary)** if all transition probabilities are independent of time.

Proposition 1.9. Let $\{X_0, X_1, \dots\}$ be a Markov chain. Then the following hold:

$$\begin{aligned} P^{(h)} &= P^h, \text{ for all } h = 1, 2, \dots \\ P_i &= P_0 \cdot P^i, \text{ for all } i = 0, 1, \dots \end{aligned} \quad (1.13)$$

Definition 1.10. Let X be a Markov chain. The vector $\pi = [\pi_1, \dots, \pi_n]$, where $\pi_k = \lim_{h \rightarrow \infty} P_h(k), k = 1, \dots, n$, (if it exists) is called a **steady-state distribution** of X .

Proposition 1.11. The steady-state distribution of a homogeneous Markov chain $X, \pi = [\pi_1, \dots, \pi_n]$, if it exists, is unique and is the solution of the $(n + 1) \times n$ linear system

$$\begin{cases} \pi P = \pi \\ \sum_{k=1}^n \pi_k = 1. \end{cases} \quad (1.14)$$

Definition 1.12. A Markov chain is called **regular** if there exists $h \geq 0$, such that

$$p_{ij}^{(h)} > 0,$$

for all $i, j = 1, \dots, n$.

Proposition 2.13. *Any regular Markov chain has a steady-state distribution.*

For more considerations on stochastic processes and Markov chains, see e.g. [2,5].

1. Counting Processes

A special case of stochastic processes are the ones where one needs to count the occurrences of some types of events over time. These are described by *counting processes*.

Definition 2.1. *A counting process $X(t), t \geq 0$, is a stochastic process that represents the number of items counted by time t .*

Counting processes deal with the number of occurrences of something over time, such as customers arriving at a supermarket, deleted errors, transmitted messages, number of job arrivals to a queue, holding times (in renewal processes), etc. In general, we refer to the occurrence of each event that is being counted as an “arrival”. Since their sample paths (values) are always non-decreasing, non-negative integers, all counting processes are discrete-state stochastic processes. They can be discrete-time or continuous-time. Next, we will consider the most widely used examples, Binomial (discrete-time) and Poisson (continuous-time) counting processes.

2.1. Binomial counting process

Consider a sequence of Bernoulli trials with probability of success p and count the number of “successes”.

Definition 2.2. *A Binomial counting process $X(n)$, is the number of successes in n Bernoulli trials, $n = 0, 1, \dots$*

Remark 2.3.

1. Obviously, a Binomial process $X(n)$ is a discrete state, discrete-time stochastic process.

2. The pdf of $X(n)$ is $B(n, p)$ at any time n . Recall that $E(X(n)) = np$.

3. The number of trials between two consecutive successes, Y , is the number of trials needed to get the next (first) success, so it has a $SGeo(p)$ pdf. Recall that $E(Y) = \frac{1}{p}$, $V(Y) = \frac{q}{p^2}$.

It is important to make the distinction between real time and the “time” variable n (“time” as in a stochastic process). Variable n is not measured in time units, it measures the number of trials. Let us assume that Bernoulli trials occur at equal time intervals, say every Δ seconds (or other time measurement units). That means that n trials occur during time $t = n\Delta$. The value of the process at time t has Binomial pdf with parameters $n = \frac{t}{\Delta}$ and p . Then the expected number of successes during t seconds is

$$E(X(n)) = E\left(X\left(\frac{t}{\Delta}\right)\right) = np = \frac{t}{\Delta}p,$$

so the expected number of successes *per second* is

$$\lambda = \frac{p}{\Delta}.$$

Definition 2.4. The quantity $\lambda = \frac{p}{\Delta}$ is called the **arrival rate**, i.e. the average number of successes per unit of time. The quantity Δ is called a **frame**, i.e. the time interval of each Bernoulli trial. The **interarrival time** is the time between successes.

We can now rephrase: p is the probability of arrival (success) during one frame (trial), $n = \frac{t}{\Delta}$ is the number of frames during time t , $X\left(\frac{t}{\Delta}\right)$ is the number of arrivals by time t .

The concepts of arrival rate and interarrival time describe arrival of jobs in discrete-time queuing systems. In such models it is assumed that no more than 1 arrival can occur during each Δ -second frame (otherwise, a smaller Δ should be considered), thus, each frame is a Bernoulli trial.

The interarrival time, Y , measured in number of frames, has a *SGeo*(p) pdf (as mentioned earlier). Since each frame takes Δ seconds, the interarrival time is $T = \Delta Y$, a rescaled *SGeo*(p) variable, whose expected value and variance are given by

$$E(T) = \Delta E(Y) = \frac{1}{\lambda}, \quad V(T) = \Delta^2 V(Y) = \frac{q}{\lambda^2}. \quad (2.1)$$

Markov property of Binomial counting processes

Obviously, the number of successes in n trials depends only on the number of successes in $n - 1$ trials (not previous values $n - 2, n - 3, \dots$), so a Binomial process has the Markov property. Thus, it is a Markov chain. Let us find the transition probability matrix. At each trial (i.e. during each frame), the number of successes $X(n)$ either increases by 1 (in case of success), or stays the same (in case of failure). Then,

$$p_{ij} = \begin{cases} p, & j = i + 1 \\ q = 1 - p, & j = i \\ 0, & \text{otherwise} \end{cases}. \quad (2.2)$$

Obviously, transition probabilities are constant over time. Hence, $X(n)$ is a stationary Markov chain with transition probability matrix given by

$$P = \begin{bmatrix} 1-p & p & 0 & \dots & 0 & \dots \\ 0 & 1-p & p & \dots & 0 & \dots \\ 0 & 0 & 1-p & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}. \quad (2.3)$$

Notice that it is an *irregular Markov* chain. Since $X(n)$ is non-decreasing, e.g. $p_{i,i-1}^{(h)} = 0$, for all $h \geq 0$ (once we have a number of success, that number can

never decrease). Hence, a Binomial counting process does not have a steady-state distribution.

Simulation of a Binomial counting process

A Binomial counting process is a sequence of Bernoulli trials. We simulate those (see e.g. [7]) and count the number of successes.

2.2 Poisson counting process

Next, we want to obtain a continuous-time counting process, where the time t runs continuously through an interval and, thus, $X(t)$ changes at infinitely many moments. This can be obtained as a limit of some discrete-time process whose frame size (time between trials) Δ approaches 0 (thus allowing more frames during any fixed period of time). We will let

$$\Delta \rightarrow 0, \text{ as } n \rightarrow \infty$$

while keeping the arrival rate $\lambda = \text{const.}$

So, let us take the limiting case of a Binomial counting process as $\Delta \rightarrow 0$. Let us consider a Binomial counting process with arrival rate of λ / time unit. $X(t)$ denotes the number of arrivals by time t .

The number of frames during time t , $n = \frac{t}{\Delta} \rightarrow \infty$, as $\Delta \rightarrow 0$.

The probability of an arrival during a frame, $p = \lambda\Delta \rightarrow 0$, as $\Delta \rightarrow 0$.

Thus, the two parameters of the Binomial pdf approach, one ∞ , the other 0, yet λ remains constant.

$X(t)$ has a $B(n, p)$ distribution with pdf $P(X = k) = C_n^k p^k (1 - p)^{n-k}$, $k = \overline{0, n}$.

Let us see what this becomes:

$$\begin{aligned} P(X = k) &= \frac{n(n-1) \dots (n-k+1)}{k!} p^k (1-p)^{n-k} \\ &= \frac{n(n-1) \dots (n-k+1)}{k!} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \\ &= \frac{(\lambda t)^k n(n-1) \dots (n-k+1)}{k! n^k} \left(1 - \frac{\lambda t}{n}\right)^{-k} \left(1 - \frac{\lambda t}{n}\right)^n \\ &\rightarrow \frac{(\lambda t)^k}{k!} \cdot 1 \cdot 1 \cdot e^{-\lambda t}, \text{ as } n \rightarrow \infty. \end{aligned}$$

So, the limiting pdf is

$$X \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \right)_{k=0,1,\dots}$$

which means $X(t)$ has a *Poiss*(λt) distribution. This is a **Poisson counting process**.

Let us analyze what happens to the other characteristics. Recall that the interarrival time $T = \Delta Y$, where Y has $SGeo(p)$ pdf. For its cdf, we have

$$\begin{aligned} F_T(t) &= P(T \leq t) = P(\Delta Y \leq n\Delta) \\ &= P(Y \leq n) = F_Y(n) \\ &= 1 - (1 - p)^n = 1 - \left(1 - \frac{\lambda t}{n}\right)^n \\ &\rightarrow 1 - e^{-\lambda t}, \text{ as } n \rightarrow \infty. \end{aligned}$$

Then its pdf is

$$f_T(t) = F'_T(t) = \lambda e^{-\lambda t}, t > 0,$$

so T has an $Exp(\lambda)$ pdf.

Simulation of a Poisson counting process

A Poisson counting process can be simulated by a special method, using the fact that each interarrival time (and the first arrival time) has an $Exp(\lambda)$ pdf, which can be easily generated using the Inverse Transform Method by $-\frac{1}{\lambda}U$, where U has a $Unif(0,1)$ pdf (see [6]).

3 Applications

Example 3.1. *Messages arrive at a communication center at the rate of 6 messages per minute. Assume arrivals of messages are modeled by a Binomial counting process.*

1. What frame size should be used to guarantee that the probability of a message arriving during each frame is 0.1?

We have $\lambda = 6/\text{min.}$ and $p = 0.1$. Thus,

$$\lambda = \frac{p}{\Delta} = \frac{1}{60} \text{ min.} = 1 \text{ sec.}$$

2. Using the chosen frame size, find the probability of no messages arriving during the next 1 minute;

So $\Delta = 1 \text{ sec.}$ In $t = 1 \text{ minute} = 60 \text{ seconds}$, there are $n = \frac{t}{\Delta} = 60$ frames. The number of messages arriving during 60 frames, $X(60)$, has a Binomial distribution with parameters $n = 60$ and $p = 0.1$. So the desired probability is

$$P(X(60) = 0) = pdf_{X(60)}(0) = 0.0018.$$

3. Using the chosen frame size, find the probability of more than 350 messages arriving during the next hour;

Similarly, in $t = 1 \text{ hour} = 3600 \text{ seconds}$, there are $n = \frac{t}{\Delta} = 3600$ frames. Thus, the number of messages arriving during one hour, $X(3600)$, has a binomial

distribution with parameters $n = 3600$ and $p = 0.1$. Then, the probability of more than 350 messages arriving during the next hour is

$$P(X(3600) > 350) = 1 - P(X(3600) \leq 350) = 1 - \text{cdf}_{X(3600)}(350) = 0.6993.$$

4. Find the average interarrival time and its standard deviation.

By (2.1) we have

$$E(T) = \frac{1}{\lambda} = \frac{1}{6} \text{ minutes} = 10 \text{ seconds},$$

$$\text{Std}(T) = \sqrt{V(T)} = \sqrt{\frac{1-p}{\lambda^2}} = \sqrt{0.0250} \text{ minutes} \approx 9.5 \text{ seconds}.$$

5. Find the probability that the next message does not arrive during the next 20 seconds.

Recall that the interarrival time $T = \Delta Y$, where Y has $S\text{Geo}(p)$ distribution and, hence $Y - 1$ has a $\text{Geo}(p)$ pdf. The next message does not arrive during the next 20 seconds, if $T > 20$. So,

$$\begin{aligned} P(T > 20) &= P(\Delta Y > 20) = P(Y > 20) \\ &= 1 - P(Y \leq 20) = 1 - P(Y - 1 \leq 19) \\ &= 1 - \text{cdf}_{Y-1}(19) = 0.1216. \end{aligned}$$

Note that this is also the probability of 0 arrivals during $n = \frac{t}{\Delta} = 20$ frames. The number of messages arriving during the next 20 seconds, $X(20)$, has a Binomial distribution with parameters $n = 20$ and $p = 0.1$. Thus, the probability that no messages arrive during the next 20 seconds is

$$P(X(20) = 0) = \text{pdf}_{X(20)}(0) = 0.1216.$$

6. The following MATLAB code gives a simulation and illustration for the number of messages arriving in one minute:

```
% Simulation of Binomial counting process with Del frame
size.
clear all
N = input('size of sample path = ');
p = input('prob. of success (arrival) = ');
Del = input('frame size (in seconds) = ');
X = zeros(1, N);
X(1) = (rand < p); % X denotes the nr. of successes
for t = 2 : N
    X(t) = X(t - 1) + (rand < p); % count the nr. of
    successes
end
clf
% illustration
axis([0 N 0 max(X)]); % allot the box for the simulated
```


2. Find the probability of more than 350 messages arriving during the next hour; Similarly, in 1 hour = 60 minutes, the number of arriving messages, $X(1)$, has a Poisson distribution with parameter $\lambda t = 360$. Then the probability of more than 350 messages arriving during the next hour is

$$P(X(60) > 350) = 1 - P(X(60) \leq 350) = 1 - cdf_{X(60)}(350) = 0.6894.$$

3. What is the average interarrival time and its standard deviation?

The interarrival time, T , now has an $Exp(\lambda) = Exp(6)$ distribution, so

$$E(T) = \frac{1}{\lambda} = \frac{1}{6} \text{ minutes} = 10 \text{ seconds},$$

$$Std(T) = \sqrt{V(T)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{6} \text{ minutes} = 10 \text{ seconds}.$$

Notice that the average interarrival time has not changed. This is to be expected, since jobs (messages) arrive at the same rate, λ , regardless of whether their arrivals are modeled by a Binomial or a Poisson process.

However, the standard deviation is slightly increased. That is because a Binomial process has a restriction on the number of arrivals during each frame, thus reducing variability.

4. Find the probability that the next message does not arrive during the next 20 seconds.

Either we work with seconds (so $\lambda = \frac{1}{10}$ /second) and compute the probability $P(T > 20)$, where T has an $Exp(1/10)$ distribution) or in minutes ($\lambda = 6$ / minute, 20 seconds = $1/3$ minutes) and compute the probability $P(T > 1/3)$, where T has an $Exp(6)$ distribution. Either way, we have

$$P(T > 20) = 1 - P(T \leq 20) = 1 - cdf_T(20) = 0.1353.$$

Again, this is the same as 0 arrivals in $1/3$ minutes, where the number of arriving messages, $X(1/3)$, has a Poisson distribution with parameter $\lambda t = 2$.

$$P\left(X\left(\frac{1}{3}\right) = 0\right) = pdf_{X\left(\frac{1}{3}\right)}(0) = 0.1353.$$

5. The following MATLAB code gives a simulation and illustration for the arrival times and the number of messages arriving in one minute:

```
% Simulation a of a Poisson process on the time interval [0,
Tmax].
clear all
lambda = input('frequency lambda = '); % given arrival rate
Tmax = input('time frame (in minutes) Tmax = '); % given time
period
arr_times = -1/lambda * log(rand); % array containing arrival
times
last_arrival = arr_times; % each interarriv. time is
Exp(lambda)
while last_arrival <= Tmax
```


Note. All computations of pdf's and cdf's were done in MATLAB.

4 Conclusions

This paper presents the main properties of counting processes. We present theoretical considerations, computational formulas and discuss properties of Binomial and Poisson counting processes. We also describe how each counting process can be simulated on the computer. As applications, we model a problem (message arrivals) by both a Binomial and a Poisson counting process, computing and discussing several quantities relating to the problem. We also illustrate the arrivals of messages in both cases using computer simulations.

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